ELSEVIER Journal of Pure and Applied Algebra 119 (1997) 47–52

JOURNAL OF PURE AND APPLIED ALGEBRA

Some combinatorial questions about polynomial mappings

Arno van den Essen [a], Vladimir Shpilrain [b,*]

[a] Department of Mathematics, University of Nijmegen, Nijmegen, Netherlands

[b] Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

Communicated by C.A. Weibel; received 25 September 1995; revised 12 February 1996

Abstract

The purpose of this note is to show how recent progress in non-commutative combinatorial algebra gives a new input to Jacobian-related problems in a commutative situation. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 13B25, 13B10, 14A05

1. Introduction

Let K be a commutative field of characteristic 0, and $P_n = K[x_1, \ldots, x_n]$ the polynomial algebra over K. We start by recalling a well-known

Jacobian Conjecture (JC). Let $p_1, \ldots, p_n \in P_n$. If the Jacobian matrix $J=(d_j(p_i))_{1 \le i,j \le n}$ is invertible, then polynomials $p_1, \ldots, p_n$ generate the whole algebra P_n.

We deliberately avoid mentioning determinant here in order to make the conjecture suitable for non-commutative algebraic systems as well (upon defining partial derivatives appropriately). A good survey on the progress made during 1939–1981 can be found in [1]. More recent survey papers are [4, 5].

Due to its obvious attractiveness, this problem has been considered by people working in different areas; in particular, several non-commutative analogs of JC appeared to be easier to settle – see e.g. [8].

In this paper, we consider polynomials which can be included in a generating set of cardinality n of the algebra P_n. We shall call those polynomials *coordinate* to simplify the language (in a non-commutative setting, one uses the notion of "primitive element" instead of coordinate).

* Corresponding author. E-mail: shpil@math.ucsb.edu.

0022-4049/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved

PII S0022-4049(96)00043-6

It is clear that every automorphism takes any coordinate polynomial of P_n to a coordinate one, so that the set of all coordinate polynomials in P_n forms an *orbit* under the action of the group $\mathrm{Aut}(P_n)$. We consider here the question of whether or not the converse is true:

Problem 1. Is it true that every endomorphism of P_n taking any coordinate polynomial to a coordinate one, is actually an automorphism?

We prove that for the algebra P_2, the answer is "yes" (here the ground field K may have an arbitrary characteristic).

Theorem 1.1. *If an endomorphism of the algebra P_2 takes every coordinate polynomial to a coordinate one, then it is actually an automorphism.*

We also note that our proof of this theorem works in the case of *free associative algebra* of rank 2 as well.

Thus, our theorem says that (in the case of P_2) if an endomorphism acts "like an automorphism" *on one particular orbit*, then it acts like an automorphism everywhere.

A more general question arises if we consider arbitrary orbits under the action of $\mathrm{Aut}(P_n)$:

Problem 2. Let p be a polynomial; $\deg p \geq 1$. Let φ be an endomorphism of P_n which preserves the orbit of p under the action of the group $\mathrm{Aut}(P_n)$. Is it true that φ is actually an automorphism?

The answer to Problem 1 is probably "yes" for any $n \geq 2$. In fact, we show below that if JC is true then the answer is indeed affirmative. Moreover, there is a "dimension shift":

Theorem 1.2. *If the Jacobian conjecture is true for the algebra P_{n-1} $(n \geq 2)$, then Problem 1 has the affirmative answer for the algebra P_n in case the ground field is algebraically closed and has characteristic zero.*

We note that Problem 1 has been settled (in the affirmative) for a free Lie algebra of an arbitrary finite rank [7], and for a free group of rank 2 (S. Ivanov, verbal communication).

Now we can go on and assume that automorphisms can be distinguished from non-automorphisms by means of their value on just a *single element*; we call it a *test element*. More formally, a polynomial $p \in P_n$ is called a test polynomial if $\varphi(p) = \alpha(p)$ for some endomorphism φ and automorphism α implies that φ is actually an automorphism itself. The condition $\varphi(p) = \alpha(p)$ can be obviously replaced here with just $\varphi(p) = p$.

For a survey on test elements in a free group, we refer to [9]. We also mention a well-known "commutator test" in a free associative algebra of rank 2 due to Dicks [3].

It is not difficult to come up with test elements in the polynomial algebra P_n over $\mathbf{R}$, the field of reals – see Example 3.1 in Section 3. On the other hand, this yields a (probably difficult)

Problem 3. Characterize test polynomials in the algebra P_n over the field $\mathbf{C}$.

We give a necessary condition for a polynomial to be a test polynomial in P_n (Proposition 3.2). This condition however is not sufficient (Example 3.3).

One might go even further on and assume that it is possible to completely determine an endomorphism by means of its value on a single element. In other words, one might look for a polynomial $p \in P_n$ with the following property: whenever $\varphi(p) = \psi(p)$ for endomorphisms φ and ψ of the algebra P_n, it follows that $\varphi = \psi$.

If φ and ψ are automorphisms, polynomials p with this property do exist – see discussion in Section 3. In contrast, we will prove:

Proposition 1.3. *Let K be a real or algebraically closed field. For any set of $n-1$ polynomials $\{p_1, \dots, p_{n-1}\} \subseteq P_n$, there exist two different endomorphisms φ, ψ of the algebra P_n, such that $\varphi(p_i) = \psi(p_i)$, for all $1 \le i \le n-1$.*

2. Problem 1: The case n = 2 and a relation to the Jacobian Conjecture

Proof of Theorem 1.1. (i) Let φ be an endomorphism of P_2 which takes every coordinate polynomial to a coordinate one. Let $\varphi(x_1) = p$ and $\varphi(x_2) = q$. Then p is a coordinate polynomial. Hence, there is an automorphism ψ such that $\psi(x_1) = p$. Therefore, on replacing φ by $\psi^{-1} \cdot \varphi$, we may assume that $p = x_1$.

(ii) Write $q = \widetilde{q} \cdot x_2 + q_0(x_1)$, $\widetilde{q} \in P_2$, $q_0(x_1) \in K[x_1]$. Since $x_2 - q_0(x_1)$ is a coordinate polynomial, so is its image under the endomorphism φ, i.e., $q - q_0(x_1)$ is a coordinate polynomial. This means $\widetilde{q} \cdot x_2$ is a coordinate polynomial and therefore irreducible in P_2. Consequently, $\widetilde{q} \in K^*$, whence $\varphi = (x_1, \widetilde{q}x_2 + q_0(x_1))$ is an automorphism. □

Remark 2.1. As we have mentioned in the Introduction, the same proof gives the same result for a free associative algebra of rank 2.

Remark 2.2. A similar proof can be carried out to get the following generalization of Theorem 1.1: if an endomorphism of the algebra P_n, $n \ge 2$, takes every coordinate $(n-1)$-tuple of polynomials to another coordinate $(n-1)$-tuple, then it is actually an automorphism.

Our proof of Theorem 1.2 is based on the following lemma which is due to Harm Derksen.

Lemma 2.3. *Let K be an algebraically closed field. Let $p_1, \dots, p_n$ be in P_n. If polynomial $\lambda_1 p_1 + \cdots + \lambda_n p_n$ has never-vanishing gradient for every non-trivial K-linear combination, then* $\det J \in K^*$, *where* $J = (d_j(p_i))_{1 \le i,j \le n}$.

Proof. If $\det J \notin K^*$, then for some $z \in K^n$, $\det J(z) = 0$. Hence, the rows of $J(z)$ are K-linearly dependent, say

$$\lambda_1(d_1(p_1)(z),\dots,d_n(p_1)(z)) + \cdots + \lambda_n(d_1(p_n)(z),\dots,d_n(p_n)(z)) = (0,\dots,0)$$

for some $\lambda_i \in K$, not all of them zero. Thus, if we put $p = \lambda_1 p_1 + \cdots + \lambda_n p_n$, then $d_1(p)(z) = \cdots = d_n(p)(z) = 0$. However, partial derivatives of p have no common zeroes by the hypothesis, hence a contradiction. □

Proof of Theorem 1.2. Let $\varphi = (p_1,\dots,p_n)$ be an endomorphism taking coordinates to coordinates. Then, arguing as in the proof of Theorem 1.1, we may assume that $p_1 = x_1$. Since each non-trivial K-linear combination of the x_i is a coordinate polynomial in P_n, we have the conditions of Lemma 2.3 satisfied. Therefore, $\det(d_j(p_i)_{1\le i,j\le n}) \in K^*$. Since $p_1 = x_1$, we deduce that $\det(d_j(p_i)_{2\le i,j\le n}) \in K^*$. Now applying the $(n-1)$-dimensional JC, we see that $K(x_1)[p_2,\dots,p_n] = K(x_1)[x_2,\dots,x_n]$, where $K(x_1)$ is the quotient field of $K[x_1]$. It follows that $K(x_1,p_2,\dots,p_n) = K(x_1,x_2,\dots,x_n)$, which by Keller's theorem [6] implies $K[x_1,p_2,\dots,p_n] = P_n$. Thus, φ is an automorphism. □

3. Test polynomials

Example 3.1. The following polynomial p is a test polynomial for distinguishing automorphisms from non-automorphisms of the algebra P_n over **R**, the field of reals, or any of its subfields:

$$p = x_1^2 + \cdots + x_n^2.$$

Indeed, suppose $\varphi(p) = p$ for some endomorphism φ of P_n. Then φ is clearly a linear mapping since monomials of highest degree in $(\varphi(x_1))^2$ cannot cancel out. Using the "chain rule", we get:

$$(x_1,\dots,x_n) = (\varphi(x_1),\dots,\varphi(x_n))J_\varphi. \tag{1}$$

If φ is a degenerate linear transformation, then the elements of the row-matrix on the right-hand side of (1) are **R**-linearly dependent, whereas the elements on the left-hand side are not. Therefore, φ must be a non-degenerate linear transformation, hence an automorphism.

Now we introduce an important notion of the *outer rank* of a polynomial.

Definition. Let $p \in P_n$ be an arbitrary polynomial. The *outer rank* of p (*orank* p) is the minimal number of generators x_i on which the image of p under an automorphism of P_n can depend.

Polynomials of maximal outer rank (i.e., of the outer rank n) appear to be of relevance to recognizing automorphisms:

Proposition 3.2. *If p is a test polynomial, then* orank $p = n$.

Proof. Suppose orank $p = m < n$. Then there is an automorphism α of the algebra P_n such that $\alpha(p) = q = q(x_1,\dots,x_m)$. Define now an endomorphism ψ as follows: $\psi(x_i) = x_i$, $1 \le i \le m$; $\psi(x_i) = 1$, $m < i \le n$. Then the endomorphism $\varphi = \alpha^{-1}\psi\alpha$ fixes p, but φ is clearly not an automorphism of P_n. □

On the other hand, we have:

Example 3.3. The polynomial $p = x_1 + x_1x_2$ has outer rank 2, but it is not a test polynomial of P_2.

Indeed, p is fixed by a non-automorphism φ which takes x_1 to $x_1 + x_1x_2$, and x_2 to 0. If the outer rank of p were equal to 1, then for some $\varphi \in \mathrm{Aut}(P_2)$, we would have $\varphi(p) = \varphi(x_1)(\varphi(x_2) + 1) \in K[x_1]$. This means that both $\varphi(x_1)$ and $\varphi(x_2)$ depend on x_1 only. A mapping like that cannot be an automorphism, hence a contradiction.

It seems plausible however that polynomials of maximal outer rank can be used as test polynomials for distinguishing automorphisms among arbitrary *monomorphisms*, i.e., injective homomorphisms.

We are now going to prove Proposition 1.3 from the Introduction. The proof is based on the following lemma.

Lemma 3.4. *Let K be a real or algebraically closed field, and let $p_1,\dots,p_{n-1}$ belong to P_n. Then the mapping $p : K^n \to K^{n-1}$ defined by $p(a) = (p_1(a),\dots,p_{n-1}(a))$ for all $a \in K^n$, is not injective.*

Proof. Let $i : K^{n-1} \to K^n$ be the natural inclusion mapping. If p is injective, then the mapping $i \circ p : K^n \to K^n$ is injective, too. Hence it is surjective by [2]. This means i is surjective, a contradiction. □

Proof of Proposition 1.3. Let $p = (p_1,\dots,p_{n-1})$ as in Lemma 3.4. Then there exist λ in K^n and α, β in K^n with $\alpha \ne \beta$ such that $p(\alpha) = \lambda = p(\beta)$. Then φ defined by $\varphi(x_i) = \alpha_i$ and ψ defined by $\psi(x_i) = \beta_i$ for all i, are as desired. □

In case when φ and ψ are automorphisms, not just arbitrary endomorphisms, polynomials with the required property do exist – a proof of this fact has been submitted to us by D. Markushevich. However, this proof appeals to several facts from algebraic geometry, and cannot be reproduced here without large amount of background material.

Acknowledgements

We are grateful to V. Lin, D. Markushevich and D. Panyushev for useful discussions. The second author also thanks the Department of Mathematics of the University of Nijmegen for warm hospitality during his visit when this work has been initiated.

References

[1] H. Bass, E.H. Connell and D. Wright, The Jacobian conjecture: reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. 7 (1982) 287–330.

[2] A. Bialynicki-Birula and M. Rosenlicht, Injective morphisms of real algebraic varieties, Proc. Amer. Math. Soc. 13 (1962) 200–203.

[3] W. Dicks, A commutator test for two elements to generate the free algebra of rank two, Bull. London Math. Soc. 14 (1982) 48–51.

[4] A. van den Essen, Polynomial maps and the Jacobian Conjecture, in: Computational Aspects of Lie Group Representations and Related Topics, Proc. 1990 Computational Algebra Seminar (CWI, Amsterdam, 1991) 29–44.

[5] A. van den Essen, Seven lectures on polynomial automorphisms, in: A. van den Essen (Ed.), Automorphisms of Affine Spaces, Proc. Conf. 'Invertible Polynomial Maps', July 1994, Curaçao (Kluwer Academic Publishers, Dordricht, 1995) 3–39.

[6] O. Keller, Ganze Cremona-Transformationen, Monatsh. Math. Phys. 47 (1939) 299–306.

[7] A.A. Mikhalev and A.A. Zolotykh, Automorphisms and primitive elements of free Lie algebras, Comm. Algebra 22 (1994) 5889–5901.

[8] V. Shpilrain, On generators of L/R^2 Lie algebras, Proc. Amer. Math. Soc. 119 (1993) 1039–1043.

[9] V. Shpilrain, Recognizing automorphisms of the free groups, Arch. Math. 62 (1994) 385–392.